

On the birth-and-assassination process, with an application to scotching a rumor in a network

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Abstract

We give new formulas on the total number of born particles in the stable birth-and-assassination process, and prove that it has a heavy-tailed distribution. We also establish that this process is a scaling limit of a process of rumor scotching in a network, and is related to a predator-prey dynamics.

Keywords: branching process, heavy tail phenomena, SIR epidemics.

MSC-class: 60J80.

1 Introduction

Birth-and-assassination process

The birth-and-assassination process was introduced by Aldous and Krebs [2], it is a variant of the branching process. The original motivation of the authors was then to analyze a scaling limit of a queueing process with blocking which appeared in database processing, see Tsitsiklis, Papadimitriou and Humblet [14]. In this paper, we show that the birth-and-assassination process exhibits some heavy-tailed distribution. For general references on heavy-tail distribution in queueing processes, see for example Mitzenmacher [9] or Resnick [12]. In this paper, we will not discuss this application. Instead, we will show that the birth-and-assassination process is also the scaling limit of a rumor spreading model which is motivated by network epidemics and dynamic data dissemination (see for example, [10], [4], [11]).

We now reproduce the formal definition of the birth-and-assassination process from [2]. Let $\mathbb{N}^f = \cup_{k=0}^{\infty} \mathbb{N}^k$ be the set of finite k-tuples of positive integers (with $N^0 = \emptyset$). Let $\{\Phi_{\mathbf{n}}\}, \mathbf{n} \in \mathbb{N}^f$, be a family of independent Poisson processes with common arrival rate λ . Let $\{K_{\mathbf{n}}\}, \mathbf{n} \in \mathbb{N}^f$, be a family of independent, identically distributed (iid), strictly positive random variables. Suppose the families $\{\Phi_{\mathbf{n}}\}$ and $\{K_{\mathbf{n}}\}$ are independent. The particle system starts at time 0 with only the ancestor particle, indexed by \emptyset . This particle produces offspring at the arrival times of Φ_{\emptyset} , which enter the system with indices (1), (2), \dots according to their birth order. Each new particle \mathbf{n} entering

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\mathcal{B} , a birth-and-assassination process with intensities $(\lambda, 1)$. As a corollary of Theorem 1, we get

Corollary 1 (Aldous and Krebs) *If $0 < \lambda < 1/4$, the process \mathcal{B} is stable. If $\lambda > 1/4$, the process \mathcal{B} is unstable.*

In the first part of this paper, we study the behavior of the process \mathcal{B} in the stable regime, especially as λ get close to $1/4$. We introduce a family of probability measures $\{P_\lambda\}$, $\lambda > 0$, on our underlying probability space such that under P_λ , \mathcal{B} is a birth-and-assassination process with intensities $(\lambda, 1)$. Let $\lambda \in (0, 1/4)$, we define N as the total number of born particles in \mathcal{B} (including the ancestor particle) and

$$\gamma(\lambda) = \sup \{u \geq 0 : E_\lambda N^u < \infty\}.$$

In particular, if $0 < \gamma(\lambda) < \infty$, from Markov Inequality, for all $0 < \epsilon < \gamma(\lambda)$, there exists a constant $C \geq 1$ such that for all $t \geq 1$,

$$P_\lambda(N > t) \leq Ct^{-\gamma(\lambda)+\epsilon}.$$

The number γ may thus be interpreted as a power tail exponent. There is a simple expression for γ .

Theorem 2 *For all $\lambda \in (0, 1/4)$,*

$$\gamma(\lambda) = \frac{1 + \sqrt{1 - 4\lambda}}{1 - \sqrt{1 - 4\lambda}}.$$

This result contrasts with the behavior of the classical branching process, where for all $\lambda < 1$: there exists a constant $c > 0$ such that $E_\lambda \exp(cN) < \infty$. This heavy tail behavior of the birth-and-assassination process is thus a striking feature of this process. Near criticality, as $\lambda \uparrow 1/4$, we get $\gamma(\lambda) \sim 1$, whereas as $\lambda \downarrow 0$, we find $\gamma(\lambda) \sim (2\lambda)^{-1}$. By recursion, we will also compute the moments of N .

Theorem 3 (i) *For all $p \geq 2$, $E_\lambda N^p < \infty$ if and only if $\lambda \in (0, p(p+1)^{-2})$.*

(ii) *If $\lambda \in (0, 1/4]$,*

$$E_\lambda N = \frac{2}{1 + \sqrt{1 - 4\lambda}}. \quad (1)$$

(iii) *If $\lambda \in (0, 2/9)$,*

$$E_\lambda N^2 = \frac{2}{3\sqrt{1 - 4\lambda} - 1}. \quad (2)$$

Theorem 3(i) is consistent with Theorem 2: $\lambda \in (0, p(p+1)^{-2})$ is equivalent to $p \in [1, (1 + \sqrt{1 - 4\lambda})(1 - \sqrt{1 - 4\lambda})^{-1})$. Theorem 3(ii) implies a surprising discontinuity of the function $\lambda \mapsto E_\lambda N$ at the critical intensity $\lambda = 1/4$: $\lim_{\lambda \uparrow 1/4} E_\lambda N = 2$. Again, this discontinuity contrasts with what happens in a standard Galton-Watson process near criticality, where for $0 < \lambda < 1$, $E_\lambda N = (1 - \lambda)^{-1}$. We will prove also that this discontinuity is specific to $\lambda = 1/4$ and for all $p \geq 2$, $\lim_{\lambda \uparrow p(p+1)^{-2}} E_\lambda[N^p] = \infty$. We will explain a method to compute all integers moments of N by recursion. The third moment has already a complicated expression (see §2.5.1). From Theorem 3(ii), we may fill the gap in Corollary 1.

Corollary 2 *If $\lambda = 1/4$, the process \mathcal{B} is stable.*

In Section 2, we will prove Theorems 2 and 3 by exhibiting a Recursive Distributional Equation (RDE) for a random variable related to N . Unfortunately, our method does not give much insights on the heavy-tail phenomena involved in the birth-and-assassination process.

Rumor scotching process

We now define the rumor scotching process on a graph. It is a nonstandard SIR dynamics (see for example [10] or [4] for some background). This process represents the dynamics of a rumor/epidemic spreading on the vertices of a graph along its edges. A vertex may be unaware of the rumor/susceptible (S), aware of the rumor and spreading it as true/infected (I), or aware of the rumor and trying to scotch it/recovered (R).

More formally, we fix a connected graph $G = (V, E)$, and let \mathcal{P}_V denote the set of subsets of V and $\mathcal{X} = (\mathcal{P}_V \times \{S, I, R\})^V$. The spread of the rumor is described by a Markov process on \mathcal{X} . For $X = (X_v)_{v \in V} \in \mathcal{X}$, with $X_v = (A_v, s_v)$, A_v is interpreted as the set of neighbors of v which can change the opinion of v on the veracity of the rumor. If $(uv) \in E$, we define the operations E_{uv} and E_v on \mathcal{X} by $(X + E_{uv})_w = (X - E_v)_w = X_w$, if $w \neq v$ and $(X + E_{uv})_v = (A_v \cup \{u\}, I)$, $(X - E_v)_v = (\emptyset, R)$. Let $\lambda > 0$ be a fixed intensity, the rumor scotching process is the Markov process with generator:

$$\begin{aligned} K(X, X + E_{uv}) &= \lambda \mathbf{1}(s_u = I) \mathbf{1}((u, v) \in E) \mathbf{1}(s_v \neq R), \\ K(X, X - E_v) &= \mathbf{1}(s_v = I) \sum_{u \in A_v} \mathbf{1}(s_u = R), \end{aligned}$$

and all other transitions have rate 0. Typically, at time 0, there is non-empty finite set of I -vertices and there is a vertex v such that A_v contains a R -vertex. The absorbing states of this process are the states without I -vertices. The case when at time 0, A_v is the set to all neighbors of v is interesting in its own (there, A_v does not evolve before $s_v = R$).

If G is the infinite k -ary tree this process has been analyzed by Kordzakhia [7] and it was defined there as the *chase-escape model*. It is thought as a predator-prey dynamics: each vertex may be unoccupied (S), occupied by a prey (I) or occupied by a predator (R). The preys spread on unoccupied vertices and predators spread on vertices occupied by preys. If G is the \mathbb{Z}^d -lattice and if there is no R -vertices, the process is the original Richardson's model [13]. With R -vertices, this process is a variant of the two-species Richardson model with prey and predators, see for example Häggström and Pemantle [6], Kordzakhia and Lalley [8]. Nothing is apparently known on this process.

In Section 3, we show that the birth-and-assassination process is the scaling limit, as n goes to infinity, of the rumor scotching process when G is the complete graph over n vertices and the intensity is λ/n (Theorem 4).

2 Integral equations for the birth-and-assassination process

2.1 Proof of Theorem 3 for the first moment

In this paragraph, we prove Theorem 3(ii). Let $X(t) \in [0, +\infty]$ be the total number of born particles in the process \mathcal{B} given that the root cannot die before time t , and $Y(t)$ be the total number of born particles given that the root dies at time t . By definition, if D is an exponential variable with mean 1 independent of Y , then $N \stackrel{d}{=} X(0) \stackrel{d}{=} Y(D)$, where the symbol $\stackrel{d}{=}$ stands for distributional equality. We notice also that the memoryless property of the exponential variable implies $X(t) \stackrel{d}{=} Y(t + D)$. The recursive structure of the birth-and-assassination process leads to the following equality in distribution

$$Y(t) \stackrel{d}{=} 1 + \sum_{i: \xi_i \leq t} X_i(t - \xi_i) \stackrel{d}{=} 1 + \sum_{i: \xi_i \leq t} X_i(\xi_i),$$

where $\Phi = \{\xi_i\}_{i \in \mathbb{N}}$ is a Poisson point process of intensity λ and $(X_i), i \in \mathbb{N}$, are independent copies of X . Note that since all variables are non-negative, there is no issue with the case $Y(t) = +\infty$. We obtain the following RDE for the random function Y :

$$Y(t) \stackrel{d}{=} 1 + \sum_{i: \xi_i \leq t} Y_i(\xi_i + D_i), \quad (3)$$

where Y_i , and D_i are independent copies of Y and D respectively. This last RDE is the cornerstone of this work.

Assuming that $E_\lambda N < \infty$ we first prove that necessarily $\lambda \in (0, 1/4)$. For convenience, we often drop the parameter λ in E_λ and other objects depending on λ . From Fubini's theorem, $EX(0) = EN = \int_0^\infty EY(t)e^{-t}dt$ and therefore $EY(t) < \infty$ for almost all $t \geq 0$. Note however that since $t \mapsto Y(t)$ is monotone for the stochastic domination, it implies that $EY(t) < \infty$ for all $t > 0$. The same argument gives the next lemma.

Lemma 1 *Let $t > 0$ and $u > 0$, if $E[N^u] < \infty$ then $E[Y(t)^u] < \infty$.*

Now, taking expectation in (3), we get

$$EY(t) = 1 + \lambda \int_0^t \int_0^\infty EY(x+s)e^{-s}dsdx.$$

Let $f_1(t) = EY(t)$, it satisfies the integral equation, for all $t \geq 0$,

$$f_1(t) = 1 + \lambda \int_0^t e^x \int_x^\infty f_1(s)e^{-s}dsdx. \quad (4)$$

Taking the derivative once and multiplying by e^{-t} , we get: $f_1'(t)e^{-t} = \lambda \int_t^\infty f_1(s)e^{-s}ds$. Then, taking the derivative a second time and multiplying by e^t : $f_1''(t) - f_1'(t) = -\lambda f_1(t)$. So, finally, f_1 solves a linear ordinary differential equation of the second order

$$x'' - x' + \lambda x = 0, \quad (5)$$

with initial condition $x(0) = 1$. If $\lambda > 1/4$ the solutions of (5) are

$$x(t) = e^{t/2}(\cos(t\sqrt{4\lambda-1}) + a\sin(t\sqrt{4\lambda-1})),$$

for some constant a . Since $f_1(t)$ is necessarily positive, this leads to a contradiction and $EN = \infty$. Assume now that $0 < \lambda < 1/4$ and let

$$\Delta = \sqrt{1-4\lambda}, \quad \alpha = \frac{1-\Delta}{2} \quad \text{and} \quad \beta = \frac{1+\Delta}{2}. \quad (6)$$

(α, β) are the roots of the polynomial $X^2 - X + \lambda = 0$. The solutions of (5) are

$$x_a(t) = (1-a)e^{\alpha t} + ae^{\beta t}$$

for some constant a . Whereas, for $\lambda = 1/4$, $\alpha = 1/2$ and the solutions of (5) are

$$x_a(t) = (at+1)e^{t/2}.$$

For $0 < \lambda \leq 1/4$, we check easily that the functions x_a with $a \geq 0$ are the nonnegative solutions of the integral equation (4).

It remains to prove that if $0 < \lambda \leq 1/4$ then $EN < \infty$ and $f_1(t) = e^{\alpha t}$. Indeed, then $EN = \int_0^\infty f_1(t)e^{-t}dt = (1-\alpha)^{-1}$ as stated in Theorem 3(ii). To this end, define $f_1^{(n)}(t) = E \min(Y(t), n)$, from (3),

$$\min(Y(t), n) \leq_{st} 1 + \sum_{i: \xi_i \leq t} \min(Y_i(\xi_i + D_i), n).$$

Taking expectation, we obtain, for all $t \geq 0$,

$$f_1^{(n)}(t) \leq 1 + \lambda \int_0^t e^x \int_x^\infty f_1^{(n)}(s)e^{-s}dsdx. \quad (7)$$

We now state a lemma which will be used multiple times in this paper. We define

$$\bar{\gamma}(\lambda) = (1+\Delta)/(1-\Delta) = \beta/\alpha. \quad (8)$$

Let $1 < u < \bar{\gamma}$ (or equivalently $\lambda < u(u+1)^{-2}$), we define \mathcal{H}_u , the set of measurable functions $h : [0, \infty) \rightarrow [0, \infty)$ such that h is non-decreasing and $\sup_{t \geq 0} h(t)e^{-u\alpha t} < \infty$. Let $C > 0$, we define the mapping from \mathcal{H}_u to \mathcal{H}_u ,

$$\Psi : h \mapsto Ce^{u\alpha t} + \lambda \int_0^t e^x \int_x^\infty h(s)e^{-s}dsdx.$$

In order to check that Ψ is indeed a mapping from \mathcal{H}_u to \mathcal{H}_u , we use the fact that if $1 < u < \bar{\gamma}$, then $u\alpha < 1$. Note also that if $1 < u < \bar{\gamma}$, then $u\alpha - \lambda - u^2\alpha^2 > 0$. If $\lambda = 1/4$, we also define the mapping from \mathcal{H}_1 to \mathcal{H}_1 ,

$$\Phi : h \mapsto 1 + \frac{1}{4} \int_0^t e^x \int_x^\infty h(s)e^{-s}dsdx.$$

(recall that for $\lambda = 1/4$, $\alpha = 1/2$).

Lemma 2 (i) Let $1 < u < \bar{\gamma}$ and $f \in \mathcal{H}_u$ such that $f \leq \Psi(f)$. Then for all $t \geq 0$,

$$f(t) \leq C \frac{u\alpha(1-u\alpha)}{u\alpha - \lambda - u^2\alpha^2} e^{u\alpha t} - C \frac{\lambda}{u\alpha - \lambda - u^2\alpha^2} e^{\alpha t}.$$

(ii) If $\lambda = 1/4$ and $f \in \mathcal{H}_1$ is such that $f \leq \Phi(f)$, then for all $t \geq 0$,

$$f(t) \leq e^{t/2}.$$

Before proving Lemma 2, we conclude the proof of Theorem 3(ii). For $0 < \lambda < 1/4$, from (7), we may apply Lemma 2(i) applied to $1 < u < \beta/\alpha$, $C = 1$. We get that

$$f_1^{(n)}(t) \leq C_u e^{\alpha u t}$$

for some $C_u > 0$. The monotone convergence theorem implies that $f_1(t) = \lim_{n \rightarrow \infty} f_1^{(n)}(t)$ exists and is bounded by $C_u e^{\alpha u t}$. Therefore f_1 solves the integral equation (4) and is equal to x_a for some $a \geq 0$. From what precedes, we get $x_a(t) \leq C_u e^{\alpha u t}$, however, since $\alpha u < \beta$, the only possibility is $a = 0$ and $f_1(t) = e^{\alpha t}$.

Similarly, if $\lambda = 1/4$, from Lemma 2(ii), $f_1(t) \leq e^{t/2}$. This proves that f_1 is finite, and we thus have $f_1 = x_a$ for some $a \geq 0$. Again, the only possibility is $a = 0$ since $x_a(t) \leq e^{t/2}$ implies $a = 0$.

Proof of Lemma 2. (i). The fixed points of the mapping Ψ are the functions $h_{a,b}$ such that

$$h_{a,b}(t) = a e^{\alpha t} + b e^{\beta t} + C \frac{u\alpha(1-u\alpha)}{u\alpha - \lambda - u^2\alpha^2} e^{u\alpha t},$$

with $a + b + C \frac{u\alpha(1-u\alpha)}{u\alpha - \lambda - u^2\alpha^2} = C$. The only fixed point in \mathcal{H}_u is $h_* := h_{a_*,0}$ with $a_* = -C\lambda/(u\alpha - \lambda - u^2\alpha^2)$. Let \mathcal{C}_u denote the set of continuous functions in \mathcal{H}_u , note that Ψ is also a mapping from \mathcal{C}_u to \mathcal{C}_u . Now let $g_0 \in \mathcal{C}_u$ and for $k \geq 1$, $g_k = \Psi(g_{k-1})$. We first prove that for all $t \geq 0$, $\lim_k g_k(t) = h_*(t)$. If $1 < u < \bar{\gamma}$ then $u\alpha(1-u\alpha) > \lambda$ and $\frac{u\alpha(1-u\alpha)}{u\alpha - \lambda - u^2\alpha^2}$ is positive. We deduce easily that if $g_0(t) \leq L e^{u\alpha t}$ then $g_1(t) = \Psi(g_0)(t) \leq C e^{u\alpha t} + \frac{L\lambda}{u\alpha(1-u\alpha)}(e^{u\alpha t} - 1) \leq L_1 e^{u\alpha t}$, with $L_1 = (C + \frac{L\lambda}{u\alpha(1-u\alpha)})$. By recursion, we obtain that $\limsup_k g_k(t) \leq L_\infty e^{u\alpha t}$, with $L_\infty = C u\alpha(1-u\alpha)/(u\alpha - \lambda - u^2\alpha^2) < \infty$. From Arzela-Ascoli's theorem, $(g_k)_{k \in \mathbb{N}}$ is relatively compact in \mathcal{C}_u and any accumulation point converges to h_* (since h_* is the only fixed point of Ψ in \mathcal{C}_u).

Now since $f \in \mathcal{H}_u$, there exists a constant $L > 0$ such that for all $t \geq 0$, $f(t) \leq g_0(t) := L e^{u\alpha t}$. The monotonicity of the mapping Ψ implies that $\Psi(f) \leq \Psi(g_0) = g_1$. By assumption, $f \leq \Psi(f)$ thus by recursion $f \leq \lim_n g_n = h_*$.

(ii). The function $x_0(t) = e^{t/2}$ is the only fixed point of Φ in \mathcal{H}_1 . Moreover, if $g(t) \leq C e^{t/2}$ then we also have $\Phi(g)(t) \leq C e^{t/2}$. Then, if g is continuous, arguing as above, from Arzela-Ascoli's theorem, $(\Phi^k(g))_{k \in \mathbb{N}}$ converges to x_0 . We conclude as in (i). \square

2.2 Proof of Theorem 3(i)

We define $f_p(t) = E_\lambda[Y(t)^p]$. As above, we often drop the parameter λ in E_λ and other objects depending on λ .

Lemma 3 *Let $p \geq 2$, there exists a polynomial Q_p with degree p such that for all $t > 0$,*

(i) *If $\lambda \in (0, p(p+1)^{-2})$, then $f_p(t) = Q_p(e^{\alpha t})$.*

(ii) *If $\lambda \geq p(p+1)^{-2}$, then $f_p(t) = \infty$,*

Note that if such polynomial Q_p exists then $Q_p(x) \geq 1$ for all $x \geq 1$. Note also that $\lambda \in (0, p(p+1)^{-2})$ implies that $p < \bar{\gamma} = \beta/\alpha$ (where $\bar{\gamma}$ was defined by (8)), and thus $p\alpha < \beta < 1$. Hence Lemma 3 implies Theorem 3(i) since $E[N^p] = \int f_p(t)e^{-t}dt$.

Let $\kappa_p(X)$ denote the p^{th} cumulant of a random variable X whose moment generating function is defined in a neighborhood of 0: $\ln Ee^{\theta X} = \sum_{p \geq 0} \kappa_p(X)\theta^p/p!$. In particular $\kappa_0(X) = 0$, $\kappa_1(X) = EX$ and $\kappa_2(X) = \text{Var}X$. Using the exponential formula

$$E \exp \sum_{\xi_i \in \Phi} h(\xi_i, Z_i) = \exp(\lambda \int_0^\infty (Ee^{h(x, Z)} - 1)dx), \quad (9)$$

valid for all non-negative function h and iid variables $(Z_i), i \in \mathbb{N}$, independent of $\Phi = \{\xi_i\}_{i \in \mathbb{N}}$ a Poisson point process of intensity λ , we obtain that for all $p \geq 1$,

$$\kappa_p \left(\sum_{i: \xi_i \leq t} h(\xi_i, Z_i) \right) = \lambda \int_0^t E h^p(x, Z) dx. \quad (10)$$

Due to this last formula, it will be easier to deal with the cumulant $g_p(t) = \kappa_p(Y(t))$. By recursion, we will prove the next lemma which implies Lemma 3.

Lemma 4 *Let $p \geq 2$, there exists a polynomial R_p with degree p , positive on $[1, \infty)$ such that, for all $t > 0$,*

(i) *If $\lambda \in (0, p(p+1)^{-2})$, then $f_p(t) < \infty$ and $g_p(t) = R_p(e^{\alpha t})$.*

(ii) *If $\lambda \geq p(p+1)^{-2}$, then $f_p(t) = \infty$,*

Proof of Lemma 4. In §2.1, we have computed f_p for $p = 1$ and found $R_1(x) = x$. Let $p \geq 2$ and assume now that the statement of the Lemma 4 holds for $q = 1, \dots, p-1$. We assume first that $f_p(t) < \infty$, we shall prove that necessarily $\lambda \in (0, p(p+1)^{-2})$ and $g_p(t) = R_p(e^{\alpha t})$. Without loss of generality we assume that $0 < \lambda < 1/4$. From Fubini's theorem, using the linearity of cumulants in (3) and (10), we get

$$\begin{aligned} g_p(t) &= \lambda \int_0^t \int_0^\infty E[Y(x+s)^p] e^{-s} ds dx \\ &= \lambda \int_0^t e^x \int_x^\infty f_p(s) e^{-s} ds dx, \end{aligned} \quad (11)$$

(note that Fubini's Theorem implies the existence of $f_p(s)$ for all $s > 0$). From Jensen inequality $f_p(t) \geq g_1(t)^p = e^{p\alpha t}$ and the integral $\int_x^\infty e^{p\alpha s} e^{-s} ds dx$ is finite if and only if $p\alpha < 1$. We may thus assume that $p\alpha < 1$. We now recall the identity: $EX^p = \sum_\pi \prod_{I \in \pi} \kappa_{|I|}(X)$, where the sum is over all set partitions of $\{1, \dots, p\}$, $I \in \pi$ means I is one of the subsets into which the set is partitioned, and $|I|$ is the cardinal

of I . This formula implies that $EX^p = \kappa_p(X) + \Sigma_{p-1}(\kappa_1(X), \dots, \kappa_{p-1}(X))$, where $\Sigma_{p-1}(x_1, \dots, x_{p-1})$ is a polynomial in $p-1$ variables with non-negative coefficients and each of its monomial $\prod_{\ell=1}^k x_{i_\ell}^{n_\ell}$ satisfies $\sum_{\ell} n_\ell i_\ell = p$. Using the recurrence hypothesis, we deduce from (11) that there exists a polynomial $\tilde{R}_p(x) = \sum_{k=1}^p r_k x^k$ of degree p with $r_p > 0$ such that

$$\begin{aligned} g_p(t) &= \lambda \int_0^t e^x \int_x^\infty \left(g_p(s) e^{-s} + \tilde{R}_p(e^{\alpha s}) e^{-s} \right) ds dx \\ &= \sum_{k=1}^p \frac{\lambda r_k}{k\alpha(1-k\alpha)} e^{k\alpha t} + \lambda \int_0^t e^x \int_x^\infty g_p(s) e^{-s} ds dx, \end{aligned} \quad (12)$$

(recall that $p\alpha < 1$). Now we take the derivative of this last expression, multiply by e^{-t} and take the derivative again. We get that g_p is a solution of the differential equation:

$$x'' - x' + \lambda x = - \sum_{k=1}^p \lambda r_k e^{k\alpha t}, \quad (13)$$

with initial condition $x(0) = 0$. Thus necessarily $g_p(t) = ae^{\alpha t} + be^{\beta t} + \varphi(t)$, where $\varphi(t)$ is a particular solution of the differential equation (13). Assume first that $\lambda \neq p(p+1)^{-2}$, then it is easy to check that $(p+1)\lambda - p\alpha$ and $p(p+1)^{-2} - \lambda$ are different from 0 and have the same sign. Looking for a function φ of the form $\varphi(t) = \sum_{k=1}^p c_k e^{k\alpha t}$ gives $c_k = -\lambda r_k (k^2 \alpha^2 - k\alpha + \lambda)^{-1} = \lambda r_k (k-1)^{-1} ((k+1)\lambda - k\alpha)^{-1}$. If $\lambda > p(p+1)^{-2}$ then $p\alpha > \beta$ and the leading term in g_p is $c_p e^{p\alpha t}$. However, if $\lambda < p(p+1)^{-2}$, $c_p < 0$ and thus $g_p(t) < 0$ for t large enough. This is a contradiction with Equation (11) which asserts that $g_p(t)$ is positive.

We now check that if $0 < \lambda < p(p+1)^{-2}$ then $f_p(t)$ is finite. We define $f_p^{(n)}(t) = E[\min(Y(t), n)^p]$. We use the following identity,

$$\left(\sum_{i=1}^N y_i \right)^p = \sum_{i=1}^N \sum_{k=0}^{p-1} \binom{p-1}{k} y_i^{k+1} \left(\sum_{j \neq i}^N y_j \right)^{p-k-1}.$$

Then from (3) we get,

$$\begin{aligned} (Y(t) - 1)^p &\stackrel{d}{=} \\ \sum_{\xi_i \leq t} Y_i(\xi_i + D_i)^p &+ \sum_{\xi_i \leq t} \sum_{k=0}^{p-2} \binom{p-1}{k} Y_i(\xi_i + D_i)^{k+1} \left(\sum_{\xi_j \neq \xi_i \leq t} Y_j(\xi_j + D_j) \right)^{p-k-1}. \end{aligned} \quad (14)$$

The recursion hypothesis implies that there exists a constant C such that $f_k(t) = Q_k(e^{\alpha t}) \leq C e^{k\alpha t}$ for all $1 \leq k \leq p-1$. Thus, the identity $Y(t)^p = (Y(t) - 1)^p - \sum_{k=0}^{p-1} \binom{p}{k} (-1)^{p-k} Y(t)^k$ gives

$$\begin{aligned} f_p^{(n)}(t) &\leq E[\min(Y(t) - 1, n)^p] + \sum_{k=0}^{p-1} \binom{p}{k} C e^{k\alpha t} \\ &\leq E[\min(Y(t) - 1, n)^p] + C_1 e^{p\alpha t}. \end{aligned}$$

From the recursion hypothesis, if $1 \leq k \leq p-1$,

$$\int_0^t \mathbb{E}[Y(x+D)^k]dx = \int_0^t e^x \int_x^\infty f_k(s)e^{-s}dsdx = \tilde{Q}_k(e^{\alpha t}) \leq Ce^{k\alpha t}$$

for some constant $C > 0$. We take the expectation in (14) and use Slyvniak's theorem to obtain

$$\begin{aligned} f_p^{(n)}(t) &\leq C_1 e^{p\alpha t} + \lambda \int_0^t e^x \int_x^\infty f_p^{(n)}(s)e^{-s}dsdx \\ &\quad + \lambda \int_0^t \sum_{k=0}^{p-2} \binom{p-1}{k} \mathbb{E}[Y_i(x+D_i)^{k+1}] \mathbb{E} \left[\left(\sum_{\xi_j \leq t} Y_j(\xi_j + D_j) \right)^{p-k-1} \right] dx \\ &\leq C_1 e^{p\alpha t} + \lambda \int_0^t e^x \int_x^\infty f_p^{(n)}(s)e^{-s}dsdx \\ &\quad + \lambda \sum_{k=0}^{p-2} \binom{p-1}{k} \tilde{Q}_{k+1}(e^{\alpha t}) \mathbb{E}[(Y(t)-1)^{p-k-1}] \\ &\leq C_2 e^{p\alpha t} + \lambda \int_0^t e^x \int_x^\infty f_p^{(n)}(s)e^{-s}dsdx \end{aligned}$$

So finally for a suitable choice of C ,

$$|f_p^{(n)}(t)| \leq Ce^{p\alpha t} + \lambda \int_0^t e^x \int_x^\infty f_p^{(n)}(s)e^{-s}dsdx. \quad (15)$$

From Lemma 2, $f_p^{(n)}(t) \leq C'e^{p\alpha t}$, and, by the monotone convergence theorem, $g_p(t) \leq f_p(t) \leq C'e^{p\alpha t}$. From what precedes: $g_p(t) = ae^{\alpha t} + be^{\beta t} + \varphi(t)$, with $\varphi(t) = \sum_{k=1}^p c_k e^{k\alpha t}$, with $c_p > 0$. If $b > 0$, since $\lambda > p(p+1)^{-2}$ then $p\alpha < \beta$ and the leading term in g_p is $be^{\beta t}$ which is in contradiction with $g_p(t) \leq C'e^{p\alpha t}$. If $b < 0$, this is a contraction with Equation (11) which asserts that $g_p(t)$ is positive. Therefore $b = 0$ and $g_p(t) = ae^{\alpha t} + \varphi(t) = R_p(e^{\alpha t})$.

It remains to check that if $\lambda = p(p+1)^{-2}$ then for all $t > 0$, $f_p(t) = \infty$. We have proved that, for all $\lambda < p(p+1)^{-2}$, $g_p(t) = u_p(\lambda)(p-1)^{-1}((p+1)\lambda - p\alpha)^{-1}e^{p\alpha t} + S_{p-1}(e^{\alpha t})$, where S_{p-1} is a polynomial of degree at most $p-1$ and $u_p(\lambda) > 0$. Note that $\lim_{\lambda \uparrow p(p+1)^{-2}} (p+1)\lambda - p\alpha = 0$. A closer look at the recursion shows also that $u_p(\lambda)$ is a sum of products of terms in λ and $\lambda(\ell-1)^{-1}((\ell+1)\lambda - \ell\alpha)^{-1}$, with $2 \leq \ell \leq p-1$. In particular, we deduce that $\lim_{\lambda \uparrow p(p+1)^{-2}} u_p(\lambda) > 0$. Similarly, the coefficients of S_{p-1} are equal to sums of products of integers and terms in λ and $\lambda(\ell-1)^{-1}((\ell+1)\lambda - \ell\alpha)^{-1}$, with $2 \leq \ell \leq p-1$. Thus they stay bounded as λ goes to $p(p+1)^{-2}$ and we obtain, for all $t > 0$,

$$\liminf_{\lambda \uparrow p(p+1)^{-2}} f_p(t) \geq \lim_{\lambda \uparrow p(p+1)^{-2}} g_p(t) = \infty. \quad (16)$$

Now, for all $t > 0$, the random variable $Y(t)$ is stochastically non-decreasing with λ . Therefore $\mathbb{E}_\lambda[Y(t)^p]$ is non-decreasing and (16) implies that $\mathbb{E}_{1/4}[Y(t)^p] = \infty$. The proof of the recursion is complete. \square

2.3 Proof of Theorem 3(iii)

In this paragraph, we prove Theorem 3(iii). Let $\lambda \in (0, 2/9)$, recall that $f_2(t) = \mathbb{E}Y(t)^2$ and $g_2(t) = \text{Var}(Y(t))$. From (11) applied to $p = 2$,

$$g_2(t) = \lambda \int_0^t \int_0^\infty g_2(x+s)e^{-s} + f_1^2(x+s)e^{-s} ds dx.$$

Since $f_1(t) = e^{\alpha t}$ and $\alpha^2 = \alpha - \lambda$, g_2 satisfies the integral equation:

$$g_2(t) = \frac{\lambda}{2(2\lambda - \alpha)} (e^{2\alpha t} - 1) + \lambda \int_0^t e^x \int_x^\infty g_2(s)e^{-s} ds dx.$$

We deduce that g_2 solves an ordinary differential equation:

$$x'' - x' + \lambda x = -\lambda e^{2\alpha t},$$

with initial condition $x(0) = 0$. Thus g_2 is of the form: $g_2(t) = ae^{\alpha t} + be^{\beta t} + \frac{\lambda}{3\lambda - 2\alpha} e^{2\alpha t}$. with $a + b + \frac{\lambda}{3\lambda - 2\alpha} = 0$. From Lemma 4, $b = 0$ so finally

$$g_2(t) = \frac{\lambda}{3\lambda - 2\alpha} (e^{2\alpha t} - e^{\alpha t}) \quad \text{and} \quad f_2(t) = 2\frac{2\lambda - \alpha}{3\lambda - 2\alpha} e^{2\alpha t} - \frac{\lambda}{3\lambda - 2\alpha} e^{\alpha t}.$$

We conclude by computing $\mathbb{E}N^2 = \int e^{-t} f_2(t) dt$.

2.4 Proof of Theorem 2

As usual we drop the parameter λ in \mathbb{E}_λ . From (8), we have $\bar{\gamma}(\lambda) = \frac{1-2\lambda+\sqrt{1-4\lambda}}{2\lambda}$. To prove Theorem 2, we shall prove two statements

$$\text{If } \mathbb{E}[N^u] < \infty \text{ then } u \leq \bar{\gamma}, \tag{17}$$

$$\text{If } 1 \leq u < \bar{\gamma} \text{ then } \mathbb{E}[N^u] < \infty. \tag{18}$$

2.4.1 Proof of (17).

Let $u \geq 1$, we assume that $\mathbb{E}[N^u] < \infty$. From Lemma 1 and (3), we get

$$\mathbb{E}[Y(t)^u] = \mathbb{E} \left(1 + \sum_{i:\xi_i \leq t} Y_i(\xi_i + D_i) \right)^u.$$

Let $f_u(t) = \mathbb{E}[Y(t)^u]$. Taking expectation and using the inequality $(x+y)^u \geq x^u + y^u$, for all positive x and y , we get:

$$\begin{aligned} f_u(t) &\geq 1 + \lambda \int_0^t \mathbb{E} f_u(x + D) dx \\ &\geq 1 + \lambda \int_0^t e^x \int_x^\infty f_u(s) e^{-s} ds dx. \end{aligned} \tag{19}$$

From Jensen's Inequality, $f_u(t) \geq f_1(t)^u = e^{u\alpha t}$. Note that the integral $\int_x^\infty e^{\alpha u s} e^{-s} ds$ is finite if and only if $u < \alpha^{-1}$. Suppose now that $\bar{\gamma} < u < \alpha^{-1}$. We use the fact: if $u > \bar{\gamma}$ then $u^2\alpha^2 - u\alpha + \lambda > 0$, to deduce that there exists $0 < \epsilon < \lambda$ such that

$$u^2\alpha^2 - u\alpha + \lambda > \epsilon. \quad (20)$$

Let $\tilde{\lambda} = \lambda - \epsilon$, $\tilde{\alpha} = \alpha(\tilde{\lambda})$, $\tilde{\beta} = \beta(\tilde{\lambda})$, we may assume that ϵ is small enough to ensure also that

$$u\alpha > \tilde{\beta}. \quad (21)$$

(Indeed, for all $\lambda \in (0, 1/4)$, $\alpha(\lambda)\bar{\gamma}(\lambda) = \beta(\lambda)$ and the mapping $\lambda \mapsto \beta(\lambda)$ is obviously continuous). We compute a lower bound from (19) as follows:

$$\begin{aligned} f_u(t) &\geq 1 + \tilde{\lambda} \int_0^t e^x \int_x^\infty f_u(s) e^{-s} ds dx + \epsilon \int_0^t e^x \int_x^\infty f_u(s) e^{-s} ds dx \\ &\geq 1 + \tilde{\lambda} \int_0^t e^x \int_x^\infty f_u(s) e^{-s} ds dx + \epsilon \int_0^t e^x \int_x^\infty e^{u\alpha s} e^{-s} ds dx \\ &\geq 1 + C(e^{u\alpha t} - 1) + \tilde{\lambda} \int_0^t e^x \int_x^\infty f_u(s) e^{-s} ds dx, \end{aligned} \quad (22)$$

with $C = \epsilon(u\alpha(1 - u\alpha))^{-1} > 0$. We consider the mapping $\Psi : h \mapsto 1 + C(e^{u\alpha t} - 1) + \tilde{\lambda} \int_0^t e^x \int_x^\infty h(s) e^{-s} ds dx$. Ψ is monotone: if for all $t \geq 0$, $h_1(t) \geq h_2(t)$ then for all $t \geq 0$, $\Psi(h_1)(t) \geq \Psi(h_2)(t)$. Since, for all $t \geq 0$, $f_u(t) \geq \Psi(f_u)(t) \geq 1$, we deduce by iteration that there exists a function h such that $h = \Psi(h) \geq 1$. Solving $h = \Psi(h)$ is simple, taking twice the derivative, we get, $h'' - h' + \tilde{\lambda}h = -\epsilon e^{p\alpha t}$. Therefore, $h = ae^{\tilde{\alpha}t} + be^{\tilde{\beta}t} - \epsilon(u^2\alpha^2 - u\alpha + \tilde{\lambda})^{-1}e^{u\alpha t}$ for some constant a and b . From (21) the leading term as t goes to infinity is equal to $-\epsilon(u^2\alpha^2 - u\alpha + \tilde{\lambda})^{-1}e^{u\alpha t}$. However from (20), $-\epsilon(u^2\alpha^2 - u\alpha + \tilde{\lambda})^{-1} < 0$ and it contradicts the assumption that $h(t) \geq 1$ for all $t \geq 0$. Therefore we have proved that $u \leq \bar{\gamma}$.

2.4.2 Proof of (18).

Let $f_u^{(n)}(t) = E[\min(Y(t), n)^u]$, we have the following lemma.

Lemma 5 *There exists a constant $C > 0$ such that for all $t \geq 0$:*

$$f_u^{(n)}(t) \leq C e^{u\alpha t} + \lambda \int_0^t e^x \int_x^\infty f_u^{(n)}(s) e^{-s} ds dx.$$

The statement (18) is a direct consequence of Lemmas 2 and 5. Indeed, note that $f_u^{(n)} \leq n^u$, thus by Lemma 2, for all $t \geq 0$, $f_u^{(n)}(t) \leq C_1 e^{u\alpha t}$ for some positive constant C_1 independent of n . From the Monotone Convergence Theorem, we deduce that, for all $t \geq 0$, $f_u(t) \leq C_1 e^{u\alpha t}$. It remains to prove Lemma 5.

Proof of Lemma 5. The lemma is already proved if u is an integer in (15). The general case is a slight extension of the same argument. We write $u = p - 1 + v$ with $v \in (0, 1)$ and $p \in \mathbb{N}^*$. We use the inequality, for all $y_i \geq 0$, $1 \leq i \leq N$,

$$\left(\sum_{i=1}^N y_i \right)^u \leq \sum_{i=1}^N \sum_{k=0}^{p-1} \binom{p-1}{k} y_i^{k+v} \left(\sum_{j \neq i}^N y_j \right)^{p-k-1}$$

(which follows from the inequality $(\sum y_i)^v \leq \sum y_i^v$). Then from (3) we get the stochastic domination

$$(Y(t) - 1)^u \leq_{st} \sum_{\xi_i \leq t} Y_i(\xi_i + D_i)^u + \sum_{\xi_i \leq t} \sum_{k=0}^{p-2} \binom{p-1}{k} Y_i(\xi_i + D_i)^{k+v} \left(\sum_{\xi_j \neq \xi_i \leq t} Y_j(\xi_j + D_j) \right)^{p-k-1}$$

From Lemma 3, there exists C such that for all $1 \leq k \leq p-1$, $f_k(t) \leq Ce^{k\alpha t}$ and $\int_0^t E[Y(x+D)^k]dx \leq Ce^{k\alpha t}$. Note also, by Jensen inequality, that for all $1 \leq k \leq p-2$, $f_{k+v}(t) \leq f_{p-1}(t)^{(k+v)/(p-1)} \leq Ce^{(k+v)\alpha t}$. The same argument (with p replaced by u) which led to (15) in the proof of Lemma 4 leads to the result. \square

2.5 Some comments on the birth-and-assassination process

2.5.1 Computation of higher moments

It is probably hard to derive an expression for all moments of N , even if in the proof of Lemma 4, we have built an expression of the cumulants of $Y(t)$ by recursion. However, exact formulas become quickly very complicated. The third moment, computed by hand, gives

$$f_3(t) = 3 \frac{3\lambda - \alpha}{4\lambda - 3\alpha} e^{3\alpha t} - 6 \frac{\lambda(2\lambda - \alpha)}{(3\lambda - 2\alpha)^2} e^{2\alpha t} + \left(1 + 6 \frac{\lambda(2\lambda - \alpha)}{(3\lambda - 2\alpha)^2} - 3 \frac{3\lambda - \alpha}{4\lambda - 3\alpha} \right) e^{\alpha t}.$$

Since $N \stackrel{d}{=} Y(D)$, we obtain,

$$EN^3 = 6 \frac{(3\lambda - \alpha)\alpha}{(4\lambda - 3\alpha)(1 - \alpha - 3\lambda)} - 6 \frac{\lambda(2\lambda - \alpha)\alpha}{(3\lambda - 2\alpha)^2(1 - \alpha - 2\lambda)} + \frac{1}{1 - \alpha}.$$

2.5.2 Integral equation of the Laplace transform

It is also possible to derive an integral equation for the Laplace transform of $Y(t)$: $L_\theta(t) = E \exp(-\theta Y(t))$, with $\theta > 0$. Indeed, using RDE (3) and the exponential formula (9),

$$\begin{aligned} L_\theta(t) &= e^{-\theta} \exp \left(\lambda \int_0^t (EL_\theta(x+D) - 1) dx \right) \\ &= e^{-\theta} \exp \left(\lambda \int_0^t e^x \int_x^\infty (L_\theta(s) - 1) e^{-s} ds dx \right). \end{aligned}$$

Taking twice the derivative, we deduce that, for all $\theta > 0$, L_θ solves the differential equation:

$$x''x - x'^2 - x'x + \lambda x^2(x-1) = 0.$$

We have not been able to use fruitfully this non-linear differential equation.

2.5.3 Probability of extinction

If $\lambda > 1/4$ from Corollary 1, the probability of extinction of \mathcal{B} is strictly less than 1. It would be very interesting to have an asymptotic formula for this probability as λ get close to $1/4$ and compare it with the Galton-Watson process. To this end, we define $\pi(t)$ as the probability of extinction of \mathcal{B} given than the root cannot die before t . With the notation of Equation (3), $\pi(t)$ satisfies

$$\pi(t) = \mathbb{E} \prod_{i: \xi_i \leq t+D} \pi(t + D - \xi_i) = \mathbb{E} \prod_{i: \xi_i \leq t+D} \pi(\xi_i),$$

Using the exponential formula (9), we find that the function π solves the integral equation:

$$\pi(t) = e^t \int_t^\infty \exp \left(-(\lambda + 1)s + \lambda \int_o^s \pi(x) dx \right) ds.$$

After a quick calculation, we deduce that π is solution of the second order non-linear differential equation

$$\frac{x' - x''}{x - x'} = \lambda(x - 1).$$

Unfortunately, we have not been able to get any result on the function $\pi(t)$ from this differential equation.

3 Rumor scotching in a complete network

3.1 Definition and result

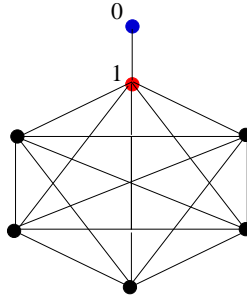


Figure 2: The graph G_6 .

We consider the rumor scotching process on the graph G_n on $\{0, \dots, n\}$ obtained by adding on the complete graph on $\{1, \dots, n\}$ the edge $(0, 1)$, see Figure 2. Let \mathcal{P}_n be the set of subsets of $\{0, \dots, n\}$. With the notation in introduction, the rumor scotching process on G_n is the Markov process on $\mathcal{X}_n = (\mathcal{P}_n \times \{S, I, R\})^n$ with generator, for

$$X = (A_i, s_i)_{0 \leq i \leq n},$$

$$K(X, X + E_{ij}) = \lambda n^{-1} \mathbf{1}(s_i = I) \mathbf{1}(s_j \neq R),$$

$$K(X, X - E_j) = \mathbf{1}(s_j = I) \left(\sum_{i=1}^n \mathbf{1}(i \in A_j) \right),$$

and all other transitions have rate 0. At time 0, the initial state is $X(0) = (X_i(0))_{0 \leq i \leq n}$ with $X_0(0) = (\emptyset, R)$, $X_1(0) = (\{0\}, I)$ and for $i \geq 2$, $X_i(0) = (\emptyset, S)$.

With this initial condition, the process describes the propagation of a rumor started from vertex 1 at time 0. After an exponential time, vertex 1 learns that the rumor is false and starts to scotch the rumor to the vertices it had previously informed. This process is a Markov process on a finite set with as absorbing states, all states without I -vertices. We define N_n as the total number of recovered vertices when the process stops evolving. We also define $Y_n(t)$ as the distribution N_n given that vertex 1 is recovered at time t . We have the following

Theorem 4 (i) *If $0 < \lambda \leq 1/4$ and $t \geq 0$, as n goes to infinity, N_n and $Y_n(t)$ converge weakly respectively to N and $Y(t)$ in the birth-and-assassination process of intensity λ .*

(ii) *If $\lambda > 1/4$, there exists $\delta > 0$ such that*

$$\liminf_n P_\lambda(N_n \geq \delta n) > 0.$$

The proof of Theorem 4 relies on the convergence of the rumor scotching process to the birth-and-assassination process, exactly as the classical SIR dynamics converges to a branching process as the size of the population goes to infinity.

3.2 Proof of Theorem 4

3.2.1 Proof of Theorem 4(i)

The proof of Theorem 4 relies on an explicit construction of the rumor scotching process. Let $(\xi_{ij}^{(n)}), 1 \leq i < j \leq n$, be a collection of independent exponential variables with parameter λn^{-1} and, for all $1 \leq i \leq j$, let D_{ij} be an independent exponential variable with parameter 1. We set $D_{ji} = D_{ij}$ and $\xi_{ji}^{(n)} = \xi_{ij}^{(n)}$. A network being a graph with marks attached on edges, we define \mathcal{K}_n as the network on the complete graph of $\{1, \dots, n\}$ where the mark attached on the edge (ij) is the pair $(\xi_{ij}^{(n)}, D_{ij})$. Now, the rumor scotching process is built on the network \mathcal{K}_n by setting $\xi_{ij}^{(n)}$ as the time for the infected particle i to infect the particle j and D_{ij} as the time for the recovered particle i to recover the particle j that it had previously infected.

The network \mathcal{K}_n has a local weak limit as n goes to infinity (see Aldous and Steele [3] for a definition of the local weak convergence). This limit network of \mathcal{K}_n is \mathcal{K} , the Poisson weighted infinite tree (PWIT) which is described as follows. The root vertex, say \emptyset , has an infinite number of children indexed by integers. The marks associated to

the edges from the root to the children are $(\xi_i, D_i)_{i \geq 1}$ where $\{\xi_i\}_{i \geq 1}$ is the realization of a Poisson process of intensity λ on \mathbb{R}_+ and $(D_i)_{i \geq 1}$ is a sequence of independent exponential variables with parameter 1. Now recursively, for each vertex $i \geq 1$ we associate an infinite number of children denoted by $(i, 1), (i, 2), \dots$ and the marks on the edges from i to its children are obtained from the realization of an independent Poisson process of intensity λ on \mathbb{R}_+ and a sequence of independent exponential variables with parameter 1. This procedure is continued for all generations. Theorem 4.1 in [3] implies the local weak convergence of \mathcal{K}_n to \mathcal{K} (for a proof see Section 3 in Aldous [1]).

Now notice that the birth-and-assassination process is the rumor scotching process on \mathcal{K} with initial condition: all vertices susceptible apart from the root which is infected and will be restored after an exponential time with mean 1.

For $s > 0$ and $\ell \in \mathbb{N}$, let $\mathcal{K}_n[s, \ell]$ be the network spanned by the set of vertices $j \in \{1, \dots, n\}$ such that there exists a sequence (i_1, \dots, i_k) with $i_1 = 1, i_k = j, k \leq \ell$ and $\max(\xi_{i_1 i_2}^{(n)}, \dots, \xi_{i_{k-1} i_k}^{(n)}) \leq s$. If τ_n is the time elapsed before an absorbing state is reached, we get that $\mathbf{1}(\tau_n \leq s) \mathbf{1}(N_n \leq \ell)$ is measurable with respect to $\mathcal{K}_n[s, \ell]$. From Theorem 4.1 in [3], we deduce that $\mathbf{1}(\tau_n \leq s) \mathbf{1}(N_n \leq \ell)$ converges in distribution to $\mathbf{1}(\tau \leq s) \mathbf{1}(N \leq \ell)$ where τ is the time elapsed before all particles die in the birth-and-assassination process. If $0 < \lambda < 1/4$, τ is almost surely finite and we deduce the statement (i).

3.2.2 Proof of Theorem 4(ii)

In order to prove part (ii) we couple the birth-and-assassination process and the rumor scotching process. We use the above notation and build the rumor scotching process on the network \mathcal{K}_n . If $X = ((A_i, s_i)_{0 \leq i \leq n}) \in \mathcal{X}_n$, we define $I(X) = \{1 \leq i \leq n : s_i = I\}$ and $S(X) = \{1 \leq i \leq n : s_i = S\}$.

Let $X = X_n(u) \in \mathcal{X}_n$ be the state of the rumor scotching process at time $u \geq 0$. Let $i \in I(X)$, we reorder the variables $(\xi_{ij}^{(n)})_{j \in S(X)}$ in non-decreasing order: $\xi_{ij_1}^{(n)} \leq \dots \leq \xi_{ij_{|S(X)|}}^{(n)}$. Define $\xi_{ij_0}^{(n)} = 0$, from the memoryless property of the exponential variable, for $1 \leq k \leq |S(X)|$, $\xi_{ij_k}^{(n)} - \xi_{ij_{k-1}}^{(n)}$ is an exponential variable with parameter $\lambda(|S(X)| - k + 1)/n$ independent of $(\xi_{ij_\ell}^{(n)} - \xi_{ij_{\ell-1}}^{(n)}, \ell < k)$. Therefore, for all $1 \leq k \leq |S(X)|$, the vector $(\xi_{ij_1}^{(n)}, \dots, \xi_{ij_k}^{(n)})$ is stochastically dominated component-wise by the vector (ξ_1, \dots, ξ_k) where $\{\xi_j\}_{j \geq 1}$ is a Poisson process of intensity $\lambda(|S(X)| - k + 1)/n$ on \mathbb{R}_+ (i.e. for all $0 \leq t_1 \leq \dots \leq t_k$, $P(\xi_{ij_1}^{(n)} \geq t_1, \dots, \xi_{ij_k}^{(n)} \geq t_k) \leq P(\xi_1 \geq t_1, \dots, \xi_k \geq t_k)$). In particular if $|S(X)| \geq (1-\delta)n$, with $0 < \delta < 1/2$, then $(\xi_{i1}^{(n)}, \dots, \xi_{i[n\delta]}^{(n)})$ is stochastically dominated component-wise by the first $[n\delta]$ arrival times of a Poisson process of intensity $\lambda(1-2\delta)$.

Now, let $\delta > 0$ such that $\lambda' = \lambda(1-2\delta) > 1/4$. We define $S_u^{(n)}, I_u^{(n)}, R_u^{(n)}$, as the number of S, I, R -particles at time $u \geq 0$ in \mathcal{K}_n , and I'_u as the number of particles "at risk" at time u in the birth-and-assassination process with intensity λ' . Let $\tau_n = \inf\{u \geq 0 : S_u^{(n)} \leq (1-\delta)n\}$. Note that if $0 \leq u \leq \tau_n$ then any I -particle has infected less than $[n\delta]$ S -particles. From what precedes, we get

$$S_u^{(n)} \mathbf{1}(u \leq \tau_n) \leq_{st} n - I'_u.$$

So that $S_u^{(n)} \leq_{st} \max(n - I'_u, (1 - \delta)n)$. In particular, since $N_n \geq \sup_{u \geq 0}(n - S_u^{(n)})$, we get

$$P_\lambda(N_n \geq \delta n) \geq P_{\lambda'}(\limsup_{u \rightarrow \infty} I'_u = \infty).$$

Finally, it is proved in [2] that if $\lambda' > 1/4$ then $P_{\lambda'}(\limsup_{u \rightarrow \infty} I'_u = \infty) > 0$.

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